Notes on trace formulas for finite groups*

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Contents

We will write down some analogs of the "global" (or Selberg) trace formula and the "global" trace formula of Arthur in the case of finite groups. We shall only assume a basic familiarity with groups and representations. For the sake of drawing an analogy, we shall often write sums as integrals.

Notation: Let G be a finite group, G_* the set of conjugacy classes, G^* the set of equivalence classes of irreducible complex representations. We denote a conjugacy class in G by

$$\{\gamma\} = \{g^{-1}\gamma g \mid g \in G\}, \qquad \gamma \in G.$$

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We often identify G_* (resp., G^*) with a complete set of representatives of these classes. Let $C(X)$ denote the vector space of complex-valued functions on a finite set X. Let dx denote the counting measure on G.

The vector space $C(G_*)$ is called the space of **class functions** on G.

1 Orbital integrals and characters

We can construct a class function from any function $f \in C(G)$ by summing over all the conjugate elements: Define the **orbital integral** by

$$\begin{array}{rcl} O & : & C(G) \to C(G_*), \\ O(f,\gamma) & = & \int_G f(x^{-1}\gamma x)\, dx, \quad \gamma \in G. \end{array}$$

Here dg denotes the counting measure. For $\pi \in G^*$, let $\mathrm{tr}\ \pi \in C(G_*)$ denote the trace (or **character**) of π. The following two lemmas are well-known (due to Frobenius or Schur):

Lemma 1. $C(G_*) = \mathrm{span}\{\mathrm{tr}\ \pi \mid \pi \in G^*\}$ *and* $\dim C(G_*) = |G_*| = |G^*|$.

In other words, the characters form a basis for the vector space of class functions.

This lemma will be referred to as the **completeness lemma**.

Lemma 2. *For $\pi, \pi' \in G^*$, we have*

$$\frac{1}{|G|}\sum_{g\in G} \mathrm{tr}\ \pi\,(g) \cdot \overline{\mathrm{tr}\ \pi'(g)} = \begin{cases} 1, & \pi \cong \pi', \\ 0, & \text{otherwise.} \end{cases} .$$

This lemma will be referred to as **orthogonality of characters**.

Corollary 3. *For all $\pi \in G^*$, we have*

$$\frac{1}{|G|}\sum_{g\in G} \mathrm{tr}\,\pi\,(g) = \begin{cases} 1, & \pi \cong 1, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

For $f_1, f_2 \in C(G)$, let

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \cdot \overline{f_2(g)}.$$

We call this the **Schur inner product** on $C(G)$. Note that the collection of functions

$$\delta_x(y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases},$$

for $x \in G$, form a basis for the vector space $C(G)$ which is orthonormal with respect to the Schur inner product. The above lemma says that the collection of irreducible characters form an orthonomal basis as well.

Example 4. *Let $G = S_3$, the symmetric group on 3 letters. Let $\sigma = (1\ 2)$, $\tau = (1\ 2\ 3)$, so*

$$G = \{1, \sigma, \tau, \tau^2, \sigma\tau, \sigma\tau^2\}.$$

Using the fact that $\sigma\tau\sigma^{-1} = \tau^2$, it can be shown that G has only 3 distinct conjugacy classes:

$$G_* = \{\{1\}, \{\sigma\}, \{\tau\}\}.$$

In fact,

$$|\{1\}| = 1, \qquad |\{\sigma\}| = 3, \qquad |\{\tau\}| = 2.$$

There are three inequivalent irreducible complex representations of G:

$$\begin{aligned} \pi_1 &: g \longmapsto \text{identity map on } \mathbb{C}(\text{trivial rep}) \\ \pi_2 &: g \longmapsto (z \longmapsto sgn(g) \cdot z) \text{ on } \mathbb{C}(\text{sign rep}) \\ \pi_3 &: g \longmapsto \text{associated } 3 \times 3 \text{ permutation matrix (standard rep)} \\ &\qquad \text{acting on } \mathbb{C}^3/\{(x,y,z) \mid x+y+z=0\}. \end{aligned}$$

Their values on the conjugacy classes are summarized as follows:

tr $\pi \backslash \|\{\gamma\}\|$	*1*	*3*	*2*
tr $\pi \backslash \{\gamma\}$	$\{1\}$	$\{\sigma\}$	$\{\tau\}$
tr π_1	*1*	*1*	*1*
tr π_2	*1*	*-1*	*1*
tr π_3	*2*	*0*	*-1*

We shall explain why the columns of this table are orthogonal in the next section.

Example 5. *The group A_5 of even permutations on $\{1,2,3,4,5\}$ has 5 conjugacy classes:*

$$\{1\}, \quad \{a=(1,2)(3,4)\}, \quad \{b=(1,2,3)\},$$

$$\{c=(1,2,3,4,5)\}, \quad \{d=(1,3,5,2,4)\}.$$

There are 5 ireducible characters. Their values on the conjugacy classes are summarized as follows:

$tr\ \pi \backslash \|\{\gamma\}\|$	1	15	20	12	12
$tr\ \pi \backslash \{\gamma\}$	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
$tr\ \pi_1$	1	1	1	1	1
$tr\ \pi_2$	3	-1	0	ϕ	$\overline{\phi}$
$tr\ \pi_3$	3	-1	0	$\overline{\phi}$	ϕ
$tr\ \pi_4$	4	0	1	-1	-1
$tr\ \pi_5$	5	1	-1	0	0

Here $\phi = (1+\sqrt{5})/2$ and $\overline{\phi} = (1-\sqrt{5})/2$.

We close this section with a

Remark 1. *One may show (without using the completeness lemma) that the orbital integral map $O : C(G) \to C(G_*)$ is surjective and that*

$$\ker\ O = \operatorname{span}\{f - f^g \mid f \in C(G),\ \ g \in G\},$$

where $f^g(x) = f(g^{-1}xg)$, for $x, g \in G$. The is equivalent to saying that the sequence

$$0 \to I(G) \stackrel{i}{\to} C(G) \stackrel{O}{\to} C(G_*) \to 0,$$

is exact, where $I(G) = C(G)^{G-1} = \{f^g - f \mid f \in C(G),\ g \in G\}$ and i is inclusion.

Since these three groups $I(G)$, $C(G)$ and $C(G_*)$ are all G-module (under the action induced by conjugation), one may ask what their cohomology groups are. Since the action of G on $C(G_*)$ is trivial and since $C(G_*)$ is a finite dimensional complex vector space, we know that $H^1(G, C(G_*)) = Hom(G, C(G_*)) = 0$. What are the other cohomology groups?

2 A Fourier expansion

Let

$$\overline{\pi}(f) = \int_G f(g)\overline{\pi(g)}\,dg = \sum_{g\in G} f(g)\overline{\pi}(g)$$

and

$$\mathrm{tr}\,\overline{\pi}\,(f) = \int_G \mathrm{tr}\,\overline{\pi}\,(g) f(g)\,dg = \sum_{g\in G} \mathrm{tr}\,\overline{\pi}\,(g) f(g).$$

By completeness, each orbital integral may be written as a linear combination of characters. Indeed, we have the following expansion.

Lemma 6. *For $f \in C(G)$ and $\gamma \in G$, we have*

$$O(f,\gamma) = \sum_{\pi\in G^*} \mathrm{tr}\,\overline{\pi}\,(f)\cdot \mathrm{tr}\,\pi\,(\gamma). \tag{2}$$

proof: By the completeness lemma, there are constants $a_\pi(f) \in \mathbb{C}$ such that

$$O(f,\gamma) = \sum_{\pi\in G^*} a_\pi(f)\cdot \mathrm{tr}\,\pi\,(\gamma).$$

Using the orthogonality of characters, we have

$$\begin{aligned}
|G|\cdot a_\pi(f) &= \int_G (\sum_{\pi'\in G^*} a_{\pi'}(f)\cdot \mathrm{tr}\,\pi'(x))\overline{\mathrm{tr}\,\pi\,(x)}\,dx \\
&= \int_G O(f,x)\overline{\mathrm{tr}\,\pi(x)}\,dx \\
&= \int_G\int_G f(g^{-1}xg)\overline{\mathrm{tr}\,\pi\,(x)}\,dgdx \\
&= \int_G\int_G f(x)\overline{\mathrm{tr}\,\pi\,(g^{-1}xg)}\,dgdx \\
&= |G|\mathrm{tr}\,\overline{\pi}\,(f).
\end{aligned}$$

□

For example, let $\gamma' \in G$ and let f denote the characteristic function of the *set* $\{\gamma'\}$. Then $\mathrm{tr}\,\pi(f) = \mathrm{tr}\,\pi(\gamma')$, In this case,

$$\sum_{\pi\in G^*} \mathrm{tr}\,\overline{\pi}(f)\cdot \mathrm{tr}\,\pi(\gamma) = \sum_{\pi\in G^*} \mathrm{tr}\,\overline{\pi}(\gamma')\cdot \mathrm{tr}\,\pi(\gamma).$$

By the lemma above, this is $= 0$ if γ is not conjugate to γ' (and is non-zero if $\gamma \in \{\gamma'\}$). This is equivalent to saying that distinct columns of the character table of a finite group are orthogonal.

3 An analog of the global trace formula

Let Γ be a subgroup of G and let G_γ denote the centralizer of γ in G:

$$G_\gamma = \{g \in G \mid g\gamma = \gamma g\}.$$

Let

$$C(\Gamma\backslash G) = \{\phi : G \to \mathbb{C} \mid \phi(hg) = \phi(g),\ \forall h \in \Gamma,\ \forall g \in G\}.$$

Note that G acts on $C(\Gamma\backslash G)$ by right translation:

$$(R(g)\phi)(x) = \phi(xg^{-1}), \quad x, g \in G.$$

(To check this we must verify that, for each $\phi \in C(\Gamma\backslash G)$, we have (a) $R(1)\phi = \phi$, which is obvious, and (b) $R(g_1)R(g_2)\phi = R(g_1g_2)\phi$, which follows from the equation

$$\begin{aligned}(R(g_1)R(g_2)\phi)(x) &= (R(g_1)\phi_{g_2})(x) = \phi_{g_2}(xg_1^{-1}) \\ &= \phi(xg_2^{-1}g_1^{-1}) = \phi(x(g_1g_2)^{-1}) = (R(g_1g_2)\phi)(x), \quad x, g_1, g_2 \in G,\end{aligned}$$

where $\phi_y(x) = \phi(xy^{-1})$.) In other words, each $g \in G$ gives rise to an automorphism

$$R(g) : C(\Gamma\backslash G) \to C(\Gamma\backslash G).$$

This is called the **right regular representation of G on $C(\Gamma\backslash G)$**. For each fixed $f \in C(G)$, define

$$R(f) : C(\Gamma\backslash G) \to C(\Gamma\backslash G)$$

by

$$(R(f)\phi)(x) = \int_G f(y)(R(y)\phi)(x)\, dy, \quad x \in G.$$

This is called the **right regular representation of** $C(G)$ **on** $C(\Gamma\backslash G)$. We may rewrite this as

$$\begin{aligned}(R(f)\phi)(x) &= \int_G f(y)(R(y)\phi)(x)\ dy \\ &= \int_G f(y)\phi(xy^{-1})\ dy \\ &= \int_G \frac{1}{\mathrm{meas}(\Gamma)} \int_\Gamma f(y)\phi(hxy^{-1})\ dh\ dy \\ &= \int_G K_f(x,y)\phi(y)\ dy,\end{aligned} \tag{3}$$

where

$$K_f(x,y) = \frac{1}{\mathrm{meas}(\Gamma)} \int_\Gamma f(y^{-1}hx)\ dh$$

and where $\mathrm{meas}(\Gamma) = \int_\Gamma 1\ dh$ is the **measure** of Γ. The function $K_f : G\times G \to \mathbb{C}$ is called the **kernel function** of the right regular representation of G on $C(\Gamma\backslash G)$.

Choose the measure dh as the counting measure, so that $\mathrm{meas}(\Gamma) = |\Gamma|$. Another way to write this is

$$K_f(x,y) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} f(y^{-1}\gamma x).$$

Lemma 7. $\mathrm{tr}(R(f)) = \frac{|\Gamma|}{|G|}\sum_{g\in G} K_f(x,x)$.

proof: The delta functions $\{\delta_x \mid x\in\Gamma\backslash G\}$, form an orthonormal basis for $C(\Gamma\backslash G)$ with respect to the inner product

$$(f_1,f_2) = \frac{|\Gamma|}{|G|}\sum_{x\in\Gamma\backslash G} f_1(x)\cdot\overline{f_2(x)}.$$

These elements may be used to compute a matrix representation of $R(f)$.

The trace is therefore given by

$$\begin{aligned}
\mathrm{tr}(R(f)) &= \sum_{g\in\Gamma\backslash G}(R(f)\delta_g,\delta_g) = \frac{|\Gamma|}{|G|}\sum_{g\in\Gamma\backslash G}\sum_{x\in\Gamma\backslash G}(R(f)\delta_g)(x)\delta_g(x) \\
&= \frac{|\Gamma|}{|G|}\sum_{g\in\Gamma\backslash G}\sum_{g\in\Gamma\backslash G}\sum_{y\in G} f(y)(R(y)\delta_g)(x)\delta_g(x) \\
&= \frac{|\Gamma|}{|G|}\sum_{g\in\Gamma\backslash G}\sum_{x\in\Gamma\backslash G}\sum_{y\in G} f(y)\delta_g(xy^{-1})\delta_g(x) \\
&= \frac{|\Gamma|}{|G|}\sum_{g\in\Gamma\backslash G}\sum_{x\in\Gamma\backslash G}\sum_{y\in g^{-1}\Gamma x} f(y)\delta_g(x) \\
&= \frac{|\Gamma|}{|G|}\sum_{g\in\Gamma\backslash G}\sum_{y\in\Gamma} f(g^{-1}hg) = \frac{1}{|G|}\sum_{g\in G}\sum_{y\in\Gamma} f(g^{-1}hg),
\end{aligned}$$

as desired. □

Lemma 8. *If*

$$R = \oplus_{\pi\in G^*} m_\pi^\Gamma \pi$$

then

(a) $\mathrm{tr}(R(f)) = \sum_{\pi\in G^*} m_\pi^\Gamma \mathrm{tr}(\pi(f))$, *for all* $f \in C(G)$,

(b) $m_\pi^\Gamma \neq 0$ *if and only if the restriction of* π *to* Γ, $res_\Gamma^G(\pi)$, *contains the trivial representation of* Γ.

proof: We only prove (b). Let $\rho \in G^*$ and $f = \mathrm{tr}(\rho)$. By (a) and orthogonality, $\mathrm{tr}(R(\mathrm{tr}(\rho))) = m_\rho$. By the previous lemma, we have

$$\mathrm{tr}(R(\mathrm{tr}(\rho))) = \frac{1}{|G|}\sum_{g\in G}\sum_{y\in\Gamma}\mathrm{tr}(\rho)(g^{-1}hg) = \sum_{y\in\Gamma}\mathrm{tr}(\rho)(h).$$

If

$$Res_\Gamma^G(\rho) = \oplus_{\sigma\in\Gamma^*} n_\sigma \sigma$$

then, by orthogonality,

$$\sum_{h\in\Gamma}\mathrm{tr}(\rho)(h) = |\Gamma| n_1,$$

where n_1 denotes the multiplicity of the trivial representation of Γ in $res_\Gamma^G(\rho)$. □

Lemma 9. *$\int_G K_f(x,x)\,dx = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma^G} |G/G_\gamma| O(f,\gamma)$, where G_γ is the centralizer of γ and where*

$$\Gamma^G = \{\{g^{-1}hg \mid g \in G\} \mid h \in \Gamma\}$$

is the set of G-conjugacy classes in Γ.

This will be called the "geometric side" of the "global" trace formula for finite groups. It is the trace of the operator $R(f)$ in (3).

proof: Since

$$\begin{aligned} G/G_\gamma &\to \{\gamma\} \\ gG_\gamma &\longmapsto g\gamma g^{-1}, \end{aligned}$$

is a bijection, we have

$$\begin{aligned}
\int_G K_f(x,x)\,dx &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\int_G f(x^{-1}\gamma x)\,dx \\
&= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma^G}\sum_{\gamma'\in\{\gamma\}}\int_G f(x^{-1}\gamma' x)\,dx \\
&= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma^G} |\{\gamma\}| O(f,\gamma) \\
&= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma^G} |G/G_\gamma| O(f,\gamma).
\end{aligned}$$

□

On the other hand, substituting the Fourier expansion (2) into the result of the above lemma, we obtain

$$\begin{aligned}
\int_G K_f(x,x)\,dx &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma^G} |G/G_\gamma| \sum_{\pi\in G^*} \mathrm{tr}\,\overline{\pi}(f)\cdot \mathrm{tr}\,\pi(\gamma) \\
&= \frac{1}{|\Gamma|}\sum_{\pi\in G^*} \mathrm{tr}\,\overline{\pi}(f) \sum_{\gamma\in\Gamma^G} |\{\gamma\}| \mathrm{tr}\,\pi(\gamma) \\
&= \frac{1}{|\Gamma|}\sum_{\pi\in G^*} \mathrm{tr}\,\overline{\pi}(f) \sum_{g\in\Gamma} \mathrm{tr}\,\pi(g).
\end{aligned}$$

This will be called the "spectral side".

Setting the geometric side equal to the spectral side gives the

Theorem 10. *("global" trace formula for finite groups) If R is the right regular representation of $C(G)$ on $C(G/\Gamma)$ then*

$$R = \oplus_{\pi \in G^*} m_\pi^\Gamma \pi$$

then, for all $f \in C(G)$, we have

$$\sum_{\pi \in G^*} m_\pi^\Gamma \mathrm{tr}(\pi)(f) = \sum_{\gamma \in \Gamma^G} |G_\gamma|^{-1} O(f,\gamma) = \frac{1}{|G|} \sum_{\pi \in G^*} \mathrm{tr}\, \overline{\pi}(f) \sum_{h \in \Gamma} \mathrm{tr}\, \pi(h).$$

3.1 Special cases of the "global" trace formula

The following result may be regarded as an analog of the Plancherel theorem.

Corollary 11. *Let $\Gamma = 1$ and let $f \in C(G)$ be arbitrary. Then*

$$f(1) = \frac{1}{|G|} \sum_{\pi \in G^*} (\dim\ \pi) \mathrm{tr}\, \overline{\pi}(f).$$

Example 12. *Assume that Γ is a subgroup of G. If π is an irreducible representation induced from Γ, say $\pi = ind_\Gamma^G \sigma$ with $\sigma \neq 1$, then the Frobenius formula states that*

$$\mathrm{tr}\, \pi(g) = \sum_{x \in G, x^{-1}gx \in \Gamma} \mathrm{tr}\, \sigma(x^{-1}gx).$$

This imples

$$\sum_{g \in \Gamma} \mathrm{tr}\, \pi(g) = \sum_{g \in \Gamma} \sum_{x \in G, x^{-1}gx \in \Gamma} \mathrm{tr}\, \sigma(x^{-1}gx).$$

Orthogonality (1) implies

$$\sum_{g \in \Gamma} \mathrm{tr}\, \sigma(x^{-1}gx) = 0,$$

since $x^{-1}\Gamma x = \Gamma$. Therefore, such induced representations do not contribute to the "trace formula" above.

Consider the case $\Gamma = G$ of the above theorem. Since $\mathrm{tr}\, \overline{1}(f) = \int_G f(g)\, dg$, the orthogonality of characters implies the following result.

Corollary 13. *(a)* $\int_G f(g)dg = \sum_{\gamma\in G_*} |G_\gamma|^{-1} O(f,\gamma)$*, for* $f \in C(G)$*.*

(b) $\int_G f(g)\, dg = \sum_{\pi\in G^*} m_\pi^G \mathrm{tr}\, \overline{\pi}(f)$*.*

We call (a) the **Weyl integration formula** for (the finite group) G. (This formula has an easier proof than the one above: as an exercise the reader may try to prove it using the decomposition $G = \bigcup_{\gamma\in G_*}\{\gamma\}$). Part (b) is another form of the Plancherel theorem.

We next consider the case where Γ is a subgroup of G of index 2 in the above theorem. In this case, Γ is automatically a normal subgroup of G. There is a theorem of Clifford which tells us precisely which representations in G^* are induced from Γ.

Lemma 14. *(Clifford) Let* $\pi \in G^*$*. Fix any* $s \in G - \Gamma$*, where* Γ *is a subgroup of* G *of index 2. Either*

(a) the restriction of π *is irreducible, in which case* π *is not induced from a representation of* Γ *but* π *is an irreducible constituent of a representation* $ind_\Gamma^G \sigma$*, for some* $\sigma \in \Gamma^*$ *which satisfies* $\sigma \cong \sigma^s$ *and the restriction of* π *to* Γ *is* σ*,*

or

(b) the restriction of π *is reducible, say* $\pi = \sigma \oplus \sigma'$ *(some* $\sigma, \sigma' \in \Gamma^*$*) in which case* $\sigma' \cong \sigma^s$*,* σ' *is inequivalent to* σ*, and*

$$\pi \cong ind_\Gamma^G \sigma \cong ind_\Gamma^G \sigma'.$$

By orthogonality, the sum $\sum_{g\in\Gamma} \mathrm{tr}\, \pi(g)$ vanishes if the restriction of π to Γ is irreducible and non-trivial. But by the remark above, the sum $\sum_{g\in\Gamma} \mathrm{tr}\, \pi(g)$ vanishes if π is induced irreducibly from Γ. By Cliffords' lemma, one of these cases must hold, so we have the following result.

Lemma 15. *Assume* Γ *is a subgroup of* G *of index* 2*. For all* $f \in C(G)$*, we have*

$$\sum_{\gamma\in\Gamma^G} |G/G_\gamma| O(f,\gamma) = \frac{1}{|\Gamma|} \sum_{\pi\in G^*, \pi|_\Gamma = 1} \mathrm{tr}\, \overline{\pi}(f).$$

4 Analog of the local trace formula for finite groups

We try to compute

$$\int_G \int_G f_1(x^{-1}yx) f_2(y)\, dx dy$$

in two ways.

Let

$$h(g) = \int_G f_1(x^{-1}gx) f_2(g)\, dx.$$

The Weyl integration formula gives

$$\begin{aligned}\int_G \int_G f_1(x^{-1}gx) f_2(g)\, dx dg &= \int_G h(g)\, dg \\ &= \sum_{\gamma \in G_*} |G_\gamma|^{-1} O(h, \gamma) \\ &= \sum_{\gamma \in G_*} |G_\gamma|^{-1} O(f_1, \gamma) O(f_2, \gamma).\end{aligned}$$

This will be called the "geometric side".

The Fourier expansion (2) gives

$$\begin{aligned}\int_G \int_G f_1(x^{-1}gx) f_2(g)\, dx dg &= \int_G O(f_1, y) f_2(y)\, dy \\ &= \sum_{\pi \in G^*} \mathrm{tr}\, \overline{\pi}(f_1) \mathrm{tr}\, \pi(f_2).\end{aligned}$$

This will be called the "spectral side".

Setting the geometric side equal to the spectral side gives the

Theorem 16. *("local" trace formula for finite groups) For $f_1, f_2 \in C(G)$, we have*

$$\sum_{\pi \in G^*} \mathrm{tr}\, \overline{\pi}(f_1) \mathrm{tr}\, \pi(f_2) = \sum_{\gamma \in G_*} |G_\gamma|^{-1} O(f_1, \gamma) O(f_2, \gamma).$$

4.1 Special cases of the local trace formula

Setting $f_1 = f_2$ gives the following curious identity:

Corollary 17. *For $f \in C(G)$ real-valued, we have*

$$\sum_{\pi \in G^*} |\mathrm{tr}\, \pi(f)|^2 = \sum_{\gamma \in G_*} |G_\gamma|^{-1} O(f, \gamma)^2.$$

Example 18. *If $G = A_5$ and f is the characteristic function of the conjugacy class $\{(1,2,3)\} \subset A_5$ then* [1] $\sum_{\pi \in G^*} |\mathrm{tr}\,\pi(f)|^2 = 400 + 400 + 400 = 1200$ *and* $\sum_{\gamma \in G_*} |G_\gamma|^{-1} O(f,\gamma)^2 = 1200$*, as expected.*

Next, we write the local trace formula for finite groups down more explicitly. First, some notation. If we let $r = |G_*| = |G^*|$ then we can let

$$\begin{aligned} G_* &= \{\gamma_1, ..., \gamma_r\}, \\ G^* &= \{\pi_1, ..., \pi_r\}, \\ G_i &= G_{\gamma_i}, \qquad 1 \leq i \leq r, \\ f_\gamma &= |G|^{-1} ch_\gamma, \qquad \gamma \in G_*, \end{aligned}$$

where ch_X denotes the characteristic function of a finite set $X \subset G$. Finally, let

$$a_{\ell m} = \mathrm{tr}\,\pi_\ell(f_{\gamma_m}), \qquad 1 \leq \ell, m \leq r.$$

(These a_{ij} are equal to $|G_j|^{-1}$ times the ij^{th} entry in the character table for G.) Then $O(\gamma_i, f_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function,

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

and where $a_{ij} = |G_j|^{-1} \mathrm{tr}\,\pi_i(\gamma_j)$.

Let $A = (a_{ij})_{1 \leq i,j \leq r} = (\vec{a}_1, ..., \vec{a}_r)$, where $\vec{a}_k$ is the $k-th$ column vector of the matrix A. If we put $f_1 = f_{\gamma_j}$ and $f_2 = f_{\gamma_i}$ then the trace formula in this case says

$$\langle \vec{a}_i, \vec{a}_j \rangle = |G_i|^{-1} \delta_{ij}, \qquad 1 \leq i, j \leq r,$$

where $\langle \vec{v}, \vec{w} \rangle = \sum_i v_i \overline{w}_i$ is the usual Hermitian inner product on $\mathbb{C}^r$. In other words, the trace formula in this case says

$$A^* A = \mathrm{diag}(|G_1|^{-1}, ..., |G_r|^{-1}),$$

where $A^* = \overline{A^t}$ is the conjugate transpose.

Example 19. *Let $G = S_3$ and let*

$$\gamma_1 = 1, \qquad \gamma_2 = (1\ 2), \qquad \gamma_3 = (1\ 2\ 3).$$

[1] This computation was performed with the aid of GAP, a group theory software package. For more details, see [J].

and we label π_1, π_2, *and* π_3 *as before. We have*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1/6 & 1/2 & 1/3 \\ 1/6 & -1/2 & 1/3 \\ 1/3 & 0 & -1/3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 3 & 2 \\ 1 & -3 & 2 \\ 2 & 0 & -2 \end{pmatrix}$$

and

$$|G_1| = 6, \qquad |G_2| = 2, \qquad |G_3| = 3.$$

It is easy to check that

$$A \cdot A^* = \mathrm{diag}(6^{-1}, 2^{-1}, 3^{-1}),$$

as predicted by the trace formula. (However, A *is not normal and* A^*A *is not diagonal.)*

Exercise 20. *Compute explicitly both sides of the equation in corollary 17 in case* $G = S_3$ *and* $f = a_1 f_{\gamma_1} + a_2 f_{\gamma_2} + a_3 f_{\gamma_3}$, *where* a_1, a_2, a_3 *are arbitrary real coefficients.*

References

[GAP] Groups, Algorithms, and Programming (GAP - a free system for computational discrete algebra), at
`http://www.gap-system.org/`

[J] D. Joyner, "Representations of A_5 using GAP," at
`http://web.usna.navy.mil/~wdj/papers.html`

[S] J.-P. Serre, **Linear representations of finite groups**, Springer-Verlag, Grad Texts in Math 42, 1977